Invariants and Monovariants

In many mathematical problems, some quantities remain unchanged or change monotonically (i.e. either increasing or decreasing). The former are called invariants while the latter are called monovariants. This concept is very useful tool in many mathematical problems. This is illustrated by the following two examples:

Example 1 (Problem C1008)

On a 8×8 chessboard with the usual colouring (black and white alternately), one may choose a pair of adjacent squares and switch the colour of each of them. Can one repeat this process so that only one black square remains on the board?

Solution

Let w be the number of white squares and b be the number of black squares.

Initially, $b = w = 32$.

Each time the process is implemented, there can be three situations.

Case 1: Both squares chosen are black.

In this case both squares will become white, so the value of b is decreased by 2.

Case 2: Only one of the chosen squares is black.

In this case, the black square becomes white and the white square becomes black. So the value of b remains unchanged.

Case 3: Booth squares chosen are white.

In this case both squares will become black, so the value of b is increased by 2.

As we can see, in each case, the parity of b remains unchanged, i.e. it is either always odd or always even.

Since the initial value of b is even, no matter how many times we implement the process, its value will still be even. Hence b can never be equal to 1. The requirement is thus not possible.

Example 2 (Problem C3001)

Five numbers, 1, 2, 3, 4, 5 are written on a board. In an operation, one may erase any two numbers, say a and b, on the board, and write the numbers $a+b$ and ab to replace them. If this process is performed repeatedly, can the numbers 12, 123, 1234, 12345 and 123456 ever appear on the board at the same time?

Solution

First, note that if both a and b are multiples of 3, then both $a+b$ and ab are multiples of 3. If one of a and b is a multiple of 3, then ab is also a multiple of 3. Hence, the number of multiples of 3 cannot decrease after an operation.

On the other hand, the number of multiples of 3 can only increase in one way, that is, when $a \equiv 1 \pmod 3$, $b \equiv 2 \pmod 3$, or vice versa, so $a+b \equiv 0 \pmod 3$ and $ab \equiv 2 \pmod 3$.

In {1, 2, 3, 4, 5}, there is one multiple of 3. In {12, 123, 1234, 12345, 123456}, there are four multiples of 3. Hence, when the number of multiples of 3 is increased to 4, the fifth number should be congruent to 2 modulo 3. However, $1234 \equiv 1 \pmod 3$. Thus we conclude that the five numbers cannot appear on the board at the same time.

Challenges

Problem C1009

The numbers 1, 2, 3, …, $2n$, where n is odd, were originally on the blackboard. Every time, one may delete two numbers a, b on the board and replace them by the number $|a-b|$. Show that the last remaining number is odd.

Problem C1010

Consider an 8×8 chessboard with the usual (alternately black and white) colouring. One may repaint all squares

(a) of a row or a column,

(b) of a 2×2 square.

Can one eventually obtain a chessboard with exactly one black square?

Problem C2010

A circle is divided into 6 equal parts, and the numbers 1, 0, 1, 0, 0, 0 are placed into the parts in order. Every time, one may choose any two adjacent parts and add 1 to both of them. Continuing this process, can we eventually obtain 6 equal numbers?

Problem C2011

On an island there are 13 blue birds, 15 white birds and 17 red birds. When two birds of different colours meet, both change into the third colour. Can all birds be changed eventually to the same colour?

Problem C2012

The following operations are permitted with the quadratic polynomial $ax^2 + bx + c$:

(a) switch a and c,

(b) replace x by $x + t$, where t is any real number.

By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

Problem C2013

The following operations are permitted with the quadratic polynomial $f(x)$:

(a) replace $f(x)$ by $x^2 f\left(\frac{1}{x} + 1\right)$,

(b) replace $f(x)$ by $(x-1)^2 f\left(\frac{1}{x-1}\right)$.

By repeating these operations, can you transform $x^2 - x - 2$ into $x^2 - x - 1$?

2.3 Starting from the set {3, 4, 12}, take any two numbers a and b from them and replace these two numbers by $0.6a - 0.8b$ and $0.8a + 0.6b$. After finitely many steps, can we obtain

(a) {4, 6, 12},

(b) the set $\{x, y, z\}$ such that $|x - 4| < \frac{1}{\sqrt{3}}, |y - 6| < \frac{1}{\sqrt{3}}$ and $|z - 12| < \frac{1}{\sqrt{3}}$.

2.4 Each of the n numbers a_1, a_2, …, a_n is equal to 1 or –1, and that $a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1 = 0$. Show that n is divisible by 4.

2.5 In the following table one may switch the signs of all numbers of a row, a column or a line parallel to one of the diagonals. Show that by repeating these operations, at least one –1 remains.

1	1	1	1
1	1	1	1
1	1	1	1
1	–1	1	1

3.1 On a 10 × 10 square board there are 100 unit squares. Nine of these unit squares are ‘infected’. In a unit time, any cells with at least two infected neighbours (we call two squares *neighbours* if they have a common side) become infected. Can the infection spread to all the unit squares eventually? How if nine is replaced by ten?

3.2 There are $\frac{n(n+1)}{2}$ coins on a desk, in the shape of an equilateral triangle with n coins on each side. Initially, all coins are turned heads up. Every time, one may turn over three coins which are mutually adjacent. Find all the values of n such that it is possible to make all the coins turned heads down.

3.3 Starting from a set of positive real numbers $S = \{a, b, c, d\}$, one replaces the set by the new set $\{|a-b|, |b-c|, |c-d|, |d-a|\}$. If this process is carried on repeatedly, can we eventually obtain the set {0, 0, 0, 0} if

(a) the numbers a, b, c and d are natural numbers?

(b) a, b, c and d are positive real numbers?

3.4 At each of the vertices of a regular pentagon, an integer is written such that the sum of the five integers exceeds zero. Every time, one may choose three adjacent integers x, y and z such that $y < 0$, and replace them by $x + y$, $-y$ and $y + z$ respectively. Show that one can repeat this process to make all the five numbers non-negative.